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Path-dependent Lagrangians in relativistic electrodynamics

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Abstract. The interaction of an electromagnetic field and a set of point charges is governed by the Maxwell–Lorentz and Minkowski equations. These relate variables—the electromagnetic field tensor and the charge–current density vector—that have a direct physical significance and are uniquely defined (once a convention for their behaviour under improper Lorentz transformations has been adopted). The Lagrangian formalism, however, requires the interaction to be expressed in terms of one variable that is uniquely defined together with one that is capable of being altered by a gauge transformation. It is demonstrated here in an explicitly Lorentz covariant manner that there exists a class of path-dependent Lagrangians with the property that the so-called 'minimal-coupling' and 'multipolar' interactions are equal. In such Lagrangians, whether the path dependence (which is a manifestation of non-uniqueness or gauge dependence) is to be carried by the electromagnetic potential or by the polarisation–magnetisation tensor of the system of particles is entirely a question of convenience. In the analysis given, the integration paths for the potential need not lie in flat hypersurfaces, and those for the polarisation–magnetisation tensor need not move with timelike velocities, as was assumed in a previous non-relativistic treatment of the subject.

1. Introduction

In the classical formulation of the electrodynamics of point charges, the interaction of field and particles is commonly introduced through the 'minimal-coupling' interaction Lagrangian, in which an electromagnetic potential is linked to the charge–current density vector. The dependence of the Lagrangian on the potential, which is susceptible to gauge transformation, can be changed into a dependence on integration paths, which are susceptible to deformation. For De Witt (1962) has shown (see also Mandelstam 1962, Belinfante 1962) that there is a class of potentials expressible as integrals of the electromagnetic field tensor along spacelike paths coming from infinity and ending at the field points. The coupling between the two parts of the system can alternatively be described by means of the 'multipolar' interaction Lagrangian, in which the electromagnetic field strengths are linked to a polarisation–magnetisation tensor and only the 'true' charge–current density to a potential. The true charges and currents represent the effect of the total charge when supposed coincident with an arbitrarily moving reference point. The multipolar Lagrangian is independent of the potential if the aggregate of particles is overall electrically neutral or if the reference point is always confined to a region (e.g. at spatial infinity) in which the potential vanishes. There still

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remains, however, the dependence of the Lagrangian on the polarisation–magnetisation tensor, as this too can be subjected to 'gauge' transformations, analogous to those of the potential. Since there exists a class of polarisation–magnetisation tensors defined as integrals along spacelike paths extending from the reference point to the charged particles (Healy 1978) or as integrals over the surfaces generated by the motion of these paths (Fiutak and Żukowski 1978), it is possible to express the multipolar Lagrangian also in an explicitly path-dependent form.

In this paper we show that the integration paths for the polarisation–magnetisation tensor are dynamically determined by the integration paths for the potential and by the motion of the charged particles. For this purpose it is necessary to modify De Witt's potential by making the integration paths start at a reference point that is not necessarily at infinity. The path-dependent minimal-coupling and multipolar Lagrangians then become identical. Moreover, gauge transformations of the potential caused by altering the integration paths induce similar 'gauge' transformations of the polarisation–magnetisation tensor. A non-relativistic treatment of this problem has been given before (see Healy 1979, where also references to previous work may be found). The present treatment is more general in that:

(i) the formalism is manifestly Lorentz covariant;

(ii) the integration paths for the potential are not confined to spacelike hyperplanes (e.g. surfaces of constant time), but can be embedded in curved spacelike hypersurfaces;

(iii) the integration paths for the polarisation–magnetisation tensor need not move with timelike velocities.

2. The charge–current density four-vector

Let the whole of the four-dimensional Minkowski space be filled with a family of distinct infinite simply-connected spacelike hypersurfaces $\sigma(\tau)$, τ being a real invariant parameter in the domain $-\infty$ to ∞. There is a unique surface passing through, and thus a unique τ associated with, any given point x. The directed surface element $d\sigma$ is a pseudovector and will be assumed to point in the direction of increasing τ in a proper (i.e. right-handed orthochronous or left-handed non-orthochronous) Lorentz frame, and in the direction of decreasing τ in an improper Lorentz frame. It is convenient to take the four-dimensional volume element d^4x and the four-dimensional delta function $\delta^4(x)$ to be pseudoscalars. We then have

$$d^4x = d\sigma^\mu \frac{\partial x_\mu}{\partial \tau} d\tau. \tag{1}$$

If the metric tensor is chosen to have signature -2, then d^4x is positive in a proper frame and negative in an improper frame.

The charged particles will be labelled by α (which is not to be summed over, unless there is an explicit indication to the contrary). Thus particle α has charge e_α, rest mass m_α and world line l_α. As this last is traced out in the positive sense, the time coordinate either strictly increases from $-\infty$ to ∞ (orthochronous frame) or strictly decreases from ∞ to $-\infty$ (non-orthochronous frame). This holds in all frames if only it holds in one, as follows from the timelike character of l_α, which represents the trajectory of a particle with non-zero rest mass. The charge–current density four-vector is defined by

$$j^\mu(x) = c \sum_\alpha e_\alpha \int_{l_\alpha} dx'^\mu \, \delta^4(x - x'), \tag{2}$$

where c is the speed of light *in vacuo*. The continuity or charge conservation equation is an immediate consequence of this definition. Since the four-dimensional delta function is a pseudoscalar, j behaves as a pseudovector, and this behaviour determines the transformation properties of the other field quantities. Thus the electromagnetic field tensor b is a pseudotensor and its dual b^* is a true tensor.

A point on l_α will be denoted by x_α. Because l_α is everywhere timelike, there is a unique point x_α on every hypersurface σ, and thus τ can be used as a parameter for l_α, i.e. x_α is a function of τ. It will be supposed that τ increases as l_α is traversed in its positive sense.

3. Lagrangian and gauge transformations

We take as a Lagrangian for the complete system the sum $\mathscr{L}$ of the free Lagrangian $\mathscr{L}_0$ given by

$$\mathscr{L}_0 = \mp \sum_\alpha m_\alpha c^2 (\dot{x}^\mu_\alpha \dot{x}_{\alpha\mu})^{1/2} - \frac{1}{16\pi} \int_\sigma b^{\mu\nu} b_{\mu\nu} \frac{\partial x_\lambda}{\partial \tau} \, \mathrm{d}\sigma^\lambda \tag{3}$$

and the interaction Lagrangian $\mathscr{L}_{\mathrm{int}}$ given by

$$\mathscr{L}_{\mathrm{int}} = -\frac{1}{c} \int_\sigma j^\mu a_\mu \frac{\partial x_\lambda}{\partial \tau} \, \mathrm{d}\sigma^\lambda. \tag{4}$$

In equation (3) the dot denotes differentiation with respect to τ, and the negative (positive) square root is to be used in a proper (improper) frame. The Lagrangian is thus a pseudoscalar. If arbitrary variations of the four-potential a are allowed, then Hamilton's principle applied to $\mathscr{L}$ leads to the Maxwell–Lorentz equations with sources. Similarly, if arbitrary variations of the particle coordinates x_α are allowed, Hamilton's principle gives the Minkowski equations of motion with the Lorentz four-force. For our purposes, however, it will be necessary to restrict the variations $\delta x_\alpha(\tau)$ to be *along* the hypersurface $\sigma(\tau)$. It is shown in the Appendix that, even with this constraint, Hamilton's principle still yields the Minkowski equations.

Although $\mathscr{L}_0$ is gauge invariant, $\mathscr{L}_{\mathrm{int}}$ is not. Under a gauge transformation with gauge function χ, $\mathscr{L}_{\mathrm{int}}$ is changed according to

$$\begin{aligned} \mathscr{L}_{\mathrm{int}} \to \tilde{\mathscr{L}}_{\mathrm{int}} &= \mathscr{L}_{\mathrm{int}} - \frac{1}{c} \int_\sigma j^\mu \, \partial_\mu \chi \frac{\partial x_\lambda}{\partial \tau} \, \mathrm{d}\sigma^\lambda \\ &= \mathscr{L}_{\mathrm{int}} - \frac{1}{c} \frac{\mathrm{d}}{\mathrm{d}\tau} \int_\sigma \chi j^\mu \, \mathrm{d}\sigma_\mu, \end{aligned} \tag{5}$$

where the second line follows from application of the four-dimensional divergence theorem (with the region of integration being bounded by the hypersurfaces $\sigma(\tau)$ and $\sigma(\tau + \mathrm{d}\tau)$) and from the continuity equation, which holds as well for the varied as for the natural motion of the system. Now $\tilde{\mathscr{L}}_{\mathrm{int}}$ depends on the new potential $\tilde{a}$ in exactly the same way that $\mathscr{L}_{\mathrm{int}}$ depended on the old potential a. Hence, if arbitrary variations of χ, and thus of $\tilde{a}$, are allowed, the new Lagrangian $\tilde{\mathscr{L}}$ must lead to the Maxwell–Lorentz and Minkowski equations (which are gauge invariant), just as did the old Lagrangian $\mathscr{L}$. However, χ may have a functional dependence on the dynamical variables that restricts its variations. (This will be true of the path-dependent gauge function to be discussed in the next section.) We may nevertheless show that, if $\chi(x)$ depends on the old potential and particle coordinates only through their values on the same hypersurface σ on which x lies, then $\tilde{\mathscr{L}}$ and $\mathscr{L}$ are equivalent, i.e. they yield equivalent equations of motion for the system. We continue to use a and the x_α as the Lagrangian coordinates and regard $\tilde{a}$ as a prescribed functional or function of these. The old and new action integrals are related by

$$\tilde{\mathscr{A}} = \mathscr{A} - \frac{1}{c}\left[\int_\sigma \chi j^\mu \, \mathrm{d}\sigma_\mu\right]_{\tau_1}^{\tau_2}. \tag{6}$$

If all variations vanish on $\sigma(\tau_1)$ and $\sigma(\tau_2)$, then $\delta\chi$ vanishes there too, by hypothesis. Moreover, $\delta j^\mu \, \mathrm{d}\sigma_\mu$ also vanishes on $\sigma(\tau_1)$ and $\sigma(\tau_2)$, provided again that $\delta x_\alpha(\tau_1) = 0 = \delta x_\alpha(\tau_2)$. This follows from the functional dependence of j on the x_α and $\dot{x}_\alpha$ and from the assumption (which is now seen to be crucial) that the variations δx_α are always along σ. Thus the variations of the action integrals $\tilde{\mathscr{A}}$ and $\mathscr{A}$ are identical and the equations of motion associated with them are equivalent. This is because the Euler–Lagrange equations are both the necessary and sufficient conditions for Hamilton's principle to hold.

4. Path-dependent four-potential

The reference point from which the integration paths are to start will be denoted by X and supposed to trace out a smooth world line L. As X may have a purely geometrical significance, L need not be everywhere timelike. It will be assumed, however, that there is a unique point X on each hypersurface σ, so that τ can be used as a parameter for L. The positive sense of L is then taken to be that of increasing τ. For each point x in $\sigma(\tau)$ we choose a curve C_x beginning at $X(\tau)$ and ending at x and lying entirely in $\sigma(\tau)$. The curves C_x are thus everywhere spacelike. Since there is a unique hypersurface $\sigma(\tau)$ passing through any given point, τ can be expressed as a function of x. Between any pair of neighbouring curves C_x there is a continuous one-to-one mapping such that the endpoints of the two curves correspond to each other. For a mapping of this kind can always be found by suitably parametrising the curves and having points with the same parameter transformed into each other. Our further results will hold for any such parametrisation.

Consider the gauge transformation for which the gauge function is minus the integral of the old potential along C_x,

$$\chi(x) = -\int_{C_x} a^\mu(x') \, \mathrm{d}x'_\mu. \tag{7}$$

Changing x to $x + \delta x$ generates a two-dimensional surface which is bounded by the closed contour formed from C_x, $C_{x+\delta x}$ (taken in its negative sense) and the straight line segments δx and $-\delta X$ joining the endpoints of the original and displaced curves (see figure 1). Calculating the change in χ by using Stokes' theorem, we obtain

$$\tilde{a}_\mu(x) \equiv a_\mu(x) + \partial_\mu \chi(x) = \frac{\partial X_\nu}{\partial x^\mu} a^\nu(X) + \int_{C_x} b^{\nu\lambda}(x') \frac{\partial x'_\nu}{\partial x^\mu} \, \mathrm{d}x'_\lambda. \tag{8}$$

This is the path-dependent four-potential. It involves the old potential a only through its value at the reference point. If this latter were always at spatial infinity (where a would be presumed to vanish), then $\tilde{a}$ would depend on the field strengths and the curves C_x alone (De Witt 1962). In the following, X need not be a point at infinity.

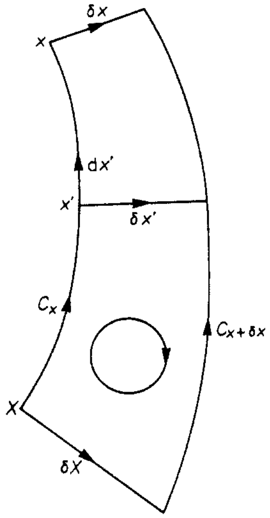

Figure 1. Illustrating the mapping $x' \to x' + \delta x'$ from the points of C_x onto those of a neighbouring curve $C_{x+\delta x}$. The surface of infinitesimal width generated by this process has surface element given by $d\sigma'_{\mu\nu} = dx'_\mu \delta x'_\nu - dx'_\nu \delta x'_\mu$ and its boundary curve has the sense indicated by the circle.

The gauge function $\chi(x)$ of equation (7) depends on the potential a only at points x' in the hypersurface σ that contains x. This is because C_x is entirely in σ. If χ is to satisfy all the conditions of § 3, then it must not depend on any particle coordinates except those in σ. This will be the case if we introduce three invariant parameters u, v and w to specify the points of σ and make the parameters of $X(\tau)$ definite functions of the parameters of the $x_\alpha(\tau)$, and then make the parameters of any point x' of C_x definite functions of the parameters of $X(\tau)$ and those of x. The functions of the parameters must be such that C_x begins at X and ends at x. If, in addition, these functions are the same for every σ, i.e. have no explicit dependence on τ, then the Lagrangian remains scleronomic after the gauge transformation.

5. Path-dependent Lagrangian—polarisation–magnetisation tensor

To investigate the polarisation–magnetisation tensor of the system, we consider a set of auxiliary curves selected from the curves C_x by the motion of the charged particles. Let C_α be that curve which, for every τ, coincides instantaneously with $C_{x_\alpha(\tau)}$. The velocity $\partial x'/\partial\tau$ of a point x' on C_α depends on the velocity $\dot{x}_\alpha$ of particle α and on the way in which C_x changes with x when $x = x_\alpha(\tau)$. As τ varies, C_α sweeps out a two-dimensional surface S_α bounded by the trajectories of particle α and the reference point (figure 2). We assume the total boundary curve to consist of the world lines l_α (positive sense) and L (negative sense) together with C_α taken in its positive sense at $\tau = -\infty$ and in its negative sense at $\tau = \infty$. If the polarisation–magnetisation tensor is defined (see also Fiutak and Żukowski 1978) by

$$m^{\mu\nu}(x) = \sum_\alpha e_\alpha \iint_{S_\alpha} \delta^4(x - x')\, d\sigma'^{\mu\nu}, \tag{9}$$

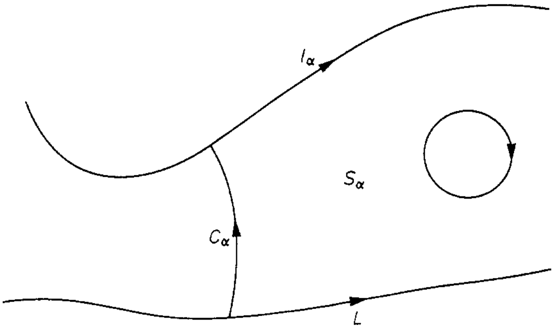

Figure 2. This diagram shows the two-dimensional surface S_α which is generated by the motion of the curve C_α and has edges along the world lines l_α and L of particle α and the reference point. The sense of the boundary curve of S_α is indicated by the circle.

then Stokes' theorem shows that the charge–current density (2) can be divided into true and bound contributions as follows:

$$j^\mu(x) = j^\mu_{\text{true}}(x) + c\,\partial_\nu m^{\mu\nu}(x). \tag{10}$$

The expression for the true charge–current density appearing here is

$$j^\mu_{\text{true}}(x) = Q\int_L \delta^4(x-x')\,\mathrm{d}x'^\mu, \tag{11}$$

Q being the total charge of the aggregate. The decomposition (10) is useful in the derivation of the covariant atomic field equations (see e.g. De Groot 1969). That there is no contribution to the line integral around the boundary of S_α from the curves C_α at $\tau = \pm\infty$ can be seen by factorising the four-dimensional delta function. We have

$$\delta^4(x) = J^{-1}\delta(\tau)\delta(u)\delta(v)\delta(w), \tag{12}$$

where δ is the ordinary one-dimensional delta function that is even under a change of sign of its argument. $J(\tau, u, v, w)$ is the Jacobian of the non-singular transformation $(\tau, u, v, w) \to (x^0, x^1, x^2, x^3)$ and is a pseudoscalar. Equation (12) can also be used to write the polarisation–magnetisation tensor (9) as a linear combination of line integrals of three-dimensional delta functions along the curves C_α (cf Healy 1978). It should be noted that the velocities $\partial x'/\partial\tau$ on C_α need not be timelike (except when $x' = x_\alpha$), unless the curves C_x are specially chosen. If this is done, then the integrals along the curves C_α are reducible, in *every* inertial frame, to instantaneous integrals along purely spatial curves (Healy 1978).

The proof of the equality of the path-dependent minimal-coupling and multipolar Lagrangians rests on the definitions of the polarisation–magnetisation tensor and of the true charge–current density given by equations (9) and (11) respectively. For these imply that

$$\tilde{\mathscr{L}}_{\text{int}} \equiv -\frac{1}{c}\int_\sigma j^\mu(x)\tilde{a}_\mu(x)\frac{\partial x_\lambda}{\partial\tau}\,\mathrm{d}\sigma^\lambda$$

$$= -\int_\sigma\left(\frac{1}{c}j^\mu_{\text{true}}(x)a_\mu(x) - \frac{1}{2}m_{\mu\nu}(x)b^{\mu\nu}(x)\right)\frac{\partial x_\lambda}{\partial\tau}\,\mathrm{d}\sigma^\lambda, \tag{13}$$

where the first expression represents the minimal-coupling and the second the multipolar form of the path-dependent interaction Lagrangian. To show the equality of these two expressions, we substitute the expansion (8) for the path-dependent potential in the first line of equation (13) and use the resolution (12) of the delta function. The equivalence of the resulting expression and that on the second line of equation (13) then follows from the definitions (9) and (11). In performing the analysis it is convenient to write the four-dimensional volume element (1) as $J\, d\tau\, du\, dv\, dw$ and to use again the factorisation (12).

The first term of the multipolar interaction Lagrangian corresponds to the coupling between the field and a single collective 'particle' of charge Q located at the reference point. This term still depends on the original potential, but is not present if either the total system is electrically neutral or the potential vanishes at the reference point. The second term is independent of the gauge of the potential, but depends on the 'gauge' of the polarisation–magnetisation tensor, i.e. on the choice of integration paths used to define it.

6. Deformation of integration paths as a gauge transformation

The gauge function that relates the potentials corresponding to different choices of the integration paths will now be examined. If X_1 and X_2 are two reference points (which may possibly coincide) and $C_x^{(1)}$ and $C_x^{(2)}$ associated integration paths (both embedded in the hypersurface σ), then equation (7) and Stokes' theorem give for the gauge function connecting the path-dependent potentials $\tilde{a}^{(1)}$ and $\tilde{a}^{(2)}$

$$\chi^{(12)}(x) \equiv \chi^{(2)}(x) - \chi^{(1)}(x) = \tfrac{1}{2} \iint_{s_x} b^{\mu\nu}(x')\, d\sigma'_{\mu\nu} + \int_{C_{12}} a^{\mu}(x')\, dx'_{\mu}, \qquad (14)$$

where C_{12} is a curve joining X_1 and X_2 and lying in σ, and s_x is a two-dimensional surface also lying in σ and bounded by C_{12}, $C_x^{(2)}$ and $C_x^{(1)}$ (taken in its negative sense). If X_1 and X_2 coincide, then $\chi^{(12)}$ does not depend on the gauge of the original potential a, as the line integral along C_{12} then vanishes (cf Belinfante 1962).

The surface integral that appears in equation (14) is independent of the surface s_x, so long as the boundary curve remains fixed. This follows from the the three-dimensional divergence theorem, since the dual tensor b^* is divergence-free. It will be useful, however, to suppose that s_x varies continuously with x, because the motion of particle α then determines a continuously moving surface s_α which coincides instantaneously with $s_{x_\alpha(\tau)}$, just as C_α coincides with $C_{x_\alpha(\tau)}$. As τ varies from $-\infty$ to ∞, the surface s_α generates a three-dimensional hypersurface σ_α bounded by $S_\alpha^{(1)}$, $S_\alpha^{(2)}$ and S_{12} (these being the two-dimensional surfaces generated by $C_\alpha^{(1)}$, $C_\alpha^{(2)}$ and C_{12}) and by s_α itself at $\tau = \pm\infty$ (see figure 3). Now corresponding to the surfaces $S_\alpha^{(1)}$ and $S_\alpha^{(2)}$ are polarisation–magnetisation tensors $m_{(1)}$ and $m_{(2)}$ defined analogously to the tensor m in equation (9). We may also define a tensor $m_{(12)}$ by

$$m_{(12)}^{\mu\nu}(x) = Q \iint_{S_{12}} \delta(x - x')\, d\sigma'^{\mu\nu}. \qquad (15)$$

The divergence of $m_{(12)}$ is $1/c$ times the difference between the true charge–current density vector associated with X_2 and that associated with X_1. If we consider only simple hypersurfaces σ_α and interchange the labels 1 and 2 if necessary, then we obtain

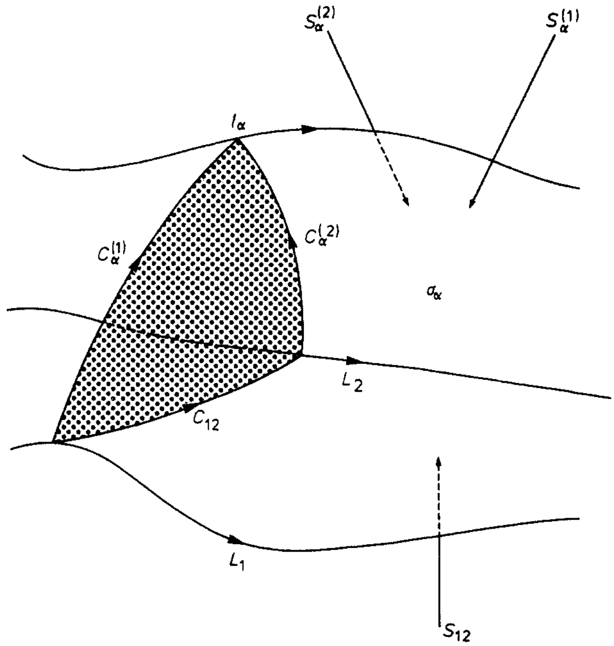

Figure 3. Showing the surface s_α (shaded) which is bounded by the curves $C_\alpha^{(1)}$, $C_\alpha^{(2)}$ and C_{12}. As these curves evolve, they generate a two-dimensional tubelike surface (consisting of $S_\alpha^{(1)}$, $S_\alpha^{(2)}$ and S_{12}) which encloses the three-dimensional hypersurface σ_α. l_α is the world line of particle α and L_1 and L_2 those of the reference points.

from the three-dimensional divergence theorem

$$m_{(2)}^{\mu\nu}(x)=m_{(1)}^{\mu\nu}(x)-m_{(12)}^{\mu\nu}(x)+\epsilon^{\mu\nu\rho\sigma}\partial_\rho v_\sigma(x), \tag{16}$$

where

$$v_\mu(x)=\sum_\alpha e_\alpha \iiint_{\sigma_\alpha} \delta(x-x')\,\mathrm{d}\sigma'_\mu \tag{17}$$

and ϵ is the Levi–Civita tensor density. Equation (16) may be regarded as a kind of gauge transformation of the polarisation–magnetisation tensor in which the vector field v plays the role of a gauge function. Under this transformation the form of the relation (10) is preserved, provided that the true charge–current density at X_1 is simultaneously shifted to X_2.

The gauge function (14) used for the transformation of the potential does not depend on the choice of the curve C_{12}. (The gauge function corresponding to a given gauge transformation is in any case unique but for an arbitrary additive constant.) This is not true of the analogous transformation for the polarisation–magnetisation tensor, however, since the 'gauge' function (17) depends on C_{12}, as does the tensor (15). Moreover, v changes when the surfaces s_α are changed, even if C_{12} is unaltered. Suppose we replace C_{12}, now called C_{12}^a, by C_{12}^b and simultaneously replace the surfaces s_x, now called s_x^a, by surfaces s_x^b. This changes the surfaces s_α^a into surfaces s_α^b. We let s^{ab} be a surface in σ bounded by C_{12}^a and C_{12}^b. (This surface vanishes if C_{12}^a and

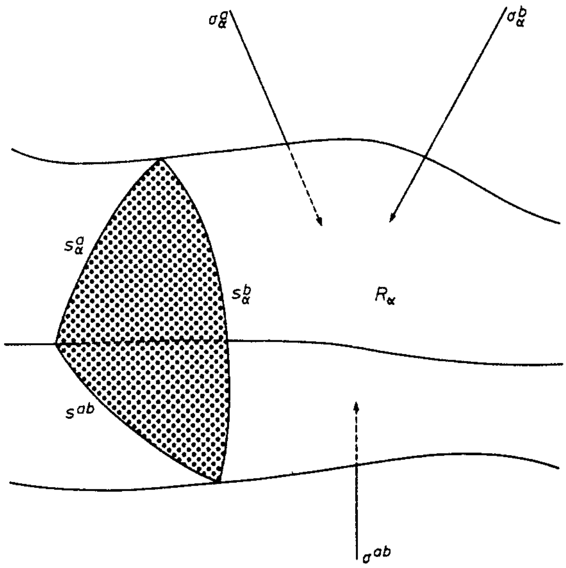

Figure 4. In this diagram the two-dimensional surfaces s_α^a, s_α^b and s^{ab} are represented by curves, while the portion of the three-dimensional hypersurface σ that is bounded by them is represented by the shaded surface. As s_α^a, s_α^b and s^{ab} evolve, they generate a tubelike three-dimensional hypersurface (consisting of σ_α^a, σ_α^b and σ^{ab}) which encloses the four-dimensional region R_α.

C_{12}^b coincide.) As τ varies, the portion of σ that is bounded by s_α^a, s_α^b and s^{ab} generates a four-dimensional region, R_α say, bounded by σ_α^a, σ_α^b and σ^{ab}, where σ^{ab} is the hypersurface generated by s^{ab} (see figure 4). We then have two vector fields v^a and v^b, defined as in equation (17), but with the integration being over σ_α^a or σ_α^b. If we treat only the case in which R_α is a simple region, and choose the sense of σ^{ab} and the order of the labels a and b properly, we obtain from the divergence theorem

$$v_\mu^b(x) = v_\mu^a(x) - v_\mu^{ab}(x) - \partial_\mu\psi(x), \tag{18}$$

where

$$v_\mu^{ab}(x) = Q \iint_{\sigma^{ab}} \delta(x-x')\,\mathrm{d}\sigma'_\mu \tag{19}$$

and

$$\psi(x) = \sum_\alpha e_\alpha \iiiint_{R_\alpha} \delta(x-x')\,\mathrm{d}^4x' = \sum_\alpha e_\alpha H_{R_\alpha}(x). \tag{20}$$

The function $H_{R_\alpha}(x)$ is zero or one according to whether x is outside or inside R_α. A further transformation is induced by altering the surface s^{ab} which is bounded by C_{12}^a and C_{12}^b. This keeps v^a and v^b fixed, but changes the scalar function ψ by changing the integration region R_α.

7. Summary

In § 2 the world lines of the charged particles were parametrised in an invariant manner by foliating the space–time manifold into spacelike hypersurfaces. The effect of gauge transformations on the Lagrangian was considered in § 3, and De Witt's gauge transformation, resulting in the path-dependent four-potential, was carried out in § 4. The path-dependent Lagrangian was shown in § 5 to lead naturally to an expression for the polarisation–magnetisation tensor. In § 6 a 'gauge' transformation of this tensor was seen to be a necessary adjunct to the gauge transformation of the potential caused by altering the integration paths. The form of Hamilton's principle that has been used involves the imposition of constraints on the particle coordinates. The associated variational problem is treated in the following Appendix by the method of Lagrangian multipliers.

Appendix

We show that in the derivation of the Euler–Lagrange equations it is sufficient to vary the particle coordinates *along* the hypersurfaces σ. If the equation of $\sigma(\tau)$ is written as $f(x, \tau) = 0$, then the action integral is

$$\mathscr{A} = \int_{\tau_1}^{\tau_2} \Big(\mathscr{L} - \sum_\alpha \lambda_\alpha(\tau) f[x_\alpha(\tau), \tau] \Big) \, \mathrm{d}\tau, \tag{A1}$$

since the subtracted sum is zero. The λ_α are, for the moment undetermined, Lagrangian multipliers. Keeping the potential a fixed, we obtain for the variation of $\mathscr{A}$

$$\delta\mathscr{A} = \sum_\alpha \int_{\tau_1}^{\tau_2} \left(\frac{\partial \mathscr{L}}{\partial x_\alpha^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial \mathscr{L}}{\partial \dot{x}_\alpha^\mu} - \lambda_\alpha \frac{\partial f}{\partial x_\alpha^\mu} \right) \delta x_\alpha^\mu \, \mathrm{d}\tau. \tag{A2}$$

It has been assumed that δx_α is along σ (and hence that $(\partial f / \partial x_\alpha^\mu) \delta x_\alpha^\mu$ is zero) and vanishes at τ_1 and τ_2. Since $\partial f / \partial x_\alpha^0$ is never zero (σ being spacelike), the variations δx_α^i can be chosen arbitrarily. Hamilton's principle then implies that the equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial \mathscr{L}}{\partial \dot{x}_\alpha^\mu} - \frac{\partial \mathscr{L}}{\partial x_\alpha^\mu} + \lambda_\alpha \frac{\partial f}{\partial x_\alpha^\mu} = 0 \tag{A3}$$

must hold for all μ, provided the λ_α have first been chosen to make them hold for $\mu = 0$. This is true even if the δx_α are further restricted by the requirement that the world lines of the varied motion, like those of the natural motion, be timelike and thus kinematically possible. Equations (A3) together with the constraint equation $f(x_\alpha, \tau) = 0$ determine the motion. Now for the Lagrangian $\mathscr{L}$ defined in § 3, the λ_α are identically zero. To prove this we note that

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial \mathscr{L}}{\partial \dot{x}_\alpha^\mu} - \frac{\partial \mathscr{L}}{\partial x_\alpha^\mu} \right) \dot{x}_\alpha^\mu = 0 \tag{A4}$$

and that, because of this equation and equations (A3),

$$\lambda_\alpha \frac{\partial f}{\partial x_\alpha^\mu} \dot{x}_\alpha^\mu = 0. \tag{A5}$$

Since both $\partial f/\partial x_\alpha$ and $\dot{x}_\alpha$ are timelike vectors, their scalar product is non-zero and the conclusion follows. Equations (A3) then reduce to the ordinary Euler–Lagrange equations, which are equivalent to the Minkowski equations of motion.

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